

# The mathematical role of (commutative and noncommutative) infinitesimal random walks over (commutative and noncommutative) riemannian manifolds in Quantum Physics

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Anderson's nonstandard construction of brownian motion as an infinitesimal random walk on the euclidean line is generalized to an Hausdorff riemannian manifold.

A nonstandard Feynman-Kac formula holding on such an Hausdorff riemannian manifold is derived.

Indications are given on how these (radically elementary) results could allow to formulate a nonstandard version of Stochastic Mechanics (avoiding both the explicitly discussed bugs of Internal Set Theory as well as the controversial renormalization of the stochastic action).

It is anyway remarked how this would contribute to hide the basic feature of Quantum Mechanics, i.e. the noncommutativity of the observables' algebra, whose structure is naturally captured in the language of Noncommutative Probability and Noncommutative Geometry.

With this respect some preliminary consideration concerning the notion of infinitesimal quantum random walk on a noncommutative riemannian manifold, the notion obtained by the Sinha-Goswami's definition of quantum brownian motion on a noncommutative riemannian manifold replacing a continuous time interval with an hyperfinite time interval, is presented.

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## I. FEYNMAN'S PHYSICAL INTUITION ABOUT PATH INTEGRALS

Let us briefly recall Feynman's way of arriving to his mathematically non rigorous but physically extraordinary path-integral formulation of Quantum Mechanics (see for instance [1], [2], [3], [4], [5]):

$$\langle q'' | \hat{U}_t | q' \rangle = \int \left\{ \begin{array}{l} q(0) = q', \\ q(t) = q'' \end{array} \right. [dq(s)] \exp(iS[q(s)]) \quad \forall q', q'' \in \mathbb{R}, \forall t \in (0, +\infty) \quad (1.1)$$

(where we adopt, from here and beyond, a unit system in which  $\hbar = 1$  and where  $S[q] := \int_0^t ds (\frac{1}{2}\dot{q}^2 - V[q(s)])$  is the non-relativistic classical action functional for a particle of unitary mass living on the real line under the influence of the conservative field force of energy potential  $V$ ) recasting it in the more usual euclidean expression:

$$\langle q'' | \hat{U}_t | q' \rangle = \int \left\{ \begin{array}{l} q(0) = q', \\ q(t) = q'' \end{array} \right. [dq(s)] \exp(-S_E[q(s)]) \quad \forall q', q'' \in \mathbb{R}, \forall t \in (0, +\infty) \quad (1.2)$$

obtained from eq.1.1 through Wick's rotation, i.e. prolonging analytically to complex values the time parameter  $t$  and computing it on the positive imaginary axis.

The basic steps are:

1. the introduction of  $n \in \mathbb{N}_+$  times:

$$t_k := k\epsilon \quad k \in \{j \in \mathbb{N} : j \leq n\} \quad (1.3)$$

where:

$$\epsilon := \frac{t}{n} \quad (1.4)$$

2. the adoption of the semi-group property of the euclidean time-evolution operator  $\hat{U}$  in order to write:

$$\langle q'' | \hat{U}_t | q' \rangle = \prod_{k=1}^{n-1} \int_{-\infty}^{+\infty} dq_k \prod_{k=1}^n \langle q_k | \hat{U}_\epsilon | q_{k-1} \rangle \quad (1.5)$$

where  $q_0 := q'$  and  $q_n := q''$ .

3. the observation that:

$$\langle q_k | \hat{U}_\epsilon | q_{k-1} \rangle = \left(\frac{1}{2\pi\epsilon}\right)^{\frac{1}{2}} \exp\left(-\frac{(q_k - q_{k-1})^2}{2\epsilon} - \epsilon V(q_k) + O(\epsilon^2)\right) \quad (1.6)$$

4. a mathematically meaningless limit for  $n \rightarrow +\infty$  (and hence  $\epsilon \rightarrow 0$ ) in which the mathematically meaningless object  $\lim_{n \rightarrow +\infty} \left(\frac{1}{2\pi\epsilon}\right)^{\frac{n}{2}} \prod_{k=1}^{n-1} dq_k$  is replaced with the mathematically meaningless functional measure  $[dq(t)]$ .

Such a derivation is as more impressive for the complete lack of mathematical meaning of its last step as well as it is impressive for the geniality of the physical intuition underlying it, a geniality resulting in the physically extraordinary path-integral formulation of Quantum Mechanics, with no doubt one of the greatest achievements of 20<sup>th</sup>-century's Theoretical Physics.

## II. THE CONDITIONAL WIENER MEASURE AND THE FEYNMAN-KAC FORMULA ON THE EUCLIDEAN LINE

Let us now briefly recall how eq. 1.2 can be recasted in a mathematically meaningful form in terms of the Wiener measure (see for instance [6], [7]).

Denoted with  $dq$  the Lebesgue measure over the measurable space  $(\mathbb{R}, \mathcal{B}(\mathbb{R}))$  (where of course  $\mathcal{B}(\mathbb{R})$  is the Borel- $\sigma$ -algebra associated to the natural topology over  $\mathbb{R}$ , namely the topology induced by the metric  $d_{euclidean}(q', q'') := |q' - q''|$ ) let us introduce the Hilbert space  $\mathcal{H} := L^2(\mathbb{R}, dq)$  and the strongly-continuous contraction semigroup (see [8], [9])  $\{T_t^{(0)} := \exp(\frac{t}{2} \frac{d^2}{dq^2})\}_{t \in (0, +\infty)}$  of operators on  $\mathcal{H}$ .

It may be easily proved that the integral kernel  $K_t^{(0)}(q', q'')$  of  $\{T_t^{(0)}\}_{t \in (0, +\infty)}$ :

$$(T_t^{(0)} f)(q') =: \int_{-\infty}^{+\infty} dq'' K_t^{(0)}(q', q'') f(q'') \quad \forall q' \in \mathbb{R}, \forall t \in (0, +\infty) \quad (2.1)$$

is given by:

$$K_t^{(0)}(q', q'') = \frac{1}{(2\pi t)^{\frac{1}{2}}} \exp\left(-\frac{(q'' - q')^2}{2t}\right) \quad \forall q', q'' \in \mathbb{R}, \forall t \in (0, +\infty) \quad (2.2)$$

Let us now observe that  $K_t^{(0)}(q', q'')$  satisfies the following basic properties:

1.

$$K_t^{(0)}(q', q'') > 0 \quad \forall q', q'' \in \mathbb{R}, \forall t \in (0, +\infty) \quad (2.3)$$

2.

$$\int_{-\infty}^{+\infty} dq'' K_t^{(0)}(q', q'') = 1 \quad \forall q' \in \mathbb{R}, \forall t \in (0, +\infty) \quad (2.4)$$

3.

$$K_{t_1+t_2}^{(0)}(q', q''') = \int_{-\infty}^{+\infty} dq'' K_{t_1}^{(0)}(q', q'') K_{t_2}^{(0)}(q'', q''') \quad \forall q', q''' \in \mathbb{R}, \forall t_1, t_2 \in (0, +\infty) \quad (2.5)$$

that guarantee that  $K_t^{(0)}(q', q'')$  is the transition probability kernel of a Markovian stochastic process  $w_t$  that one defines as the brownian motion over the metric space  $(\mathbb{R}, d_{euclidean})$ .

Introduced furthermore the functional space:

$$\mathcal{C}(q', 0; q'', t) := \{q : [0, t] \mapsto \mathbb{R} \text{ } d_{euclidean} - \text{continuous} : q(0) = q', q(t) = q''\} \quad q', q'' \in \mathbb{R}, t \in (0, +\infty) \quad (2.6)$$

<sup>1</sup> and the cylinder sets of  $\mathcal{C}(q', 0; q'', t)$ :

$$\begin{aligned} \Gamma(q', 0; \{B_k; t_k\}_{k=1}^{n-1}; q'', t) &:= \{q \in \mathcal{C}(q', 0; q'', t) : q(t_k) \in B_k \\ &\forall k \in \{j \in \mathbb{N} : 1 \leq j \leq n-1\}\} \quad 0 < \dots < t_k < t_{k+1} < \dots < t, B_k \in \mathcal{B}(\mathbb{R}) \forall k \in \{j \in \mathbb{N} : 1 \leq j \leq n-1\}, n \in \mathbb{N}_+ \end{aligned} \quad (2.7)$$

one defines the value of the conditional Wiener measure on cylinder sets as:

$$W[\Gamma(q', 0; \{B_k; t_k\}_{k=1}^{n-1}; q'', t)] := \prod_{k=1}^{n-1} \int_{B_k} dq_k \prod_{k=1}^n K_{t_k - t_{k-1}}^{(0)}(q_{k-1}, q_k) \quad (2.8)$$

where  $q_0 := q'$  and  $q_n := q''$ .

Then it may be proved that:

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<sup>1</sup> where  $d_{euclidean}$ -continuity, i.e continuity w.r.t. to the Borel  $\sigma$ -algebra associated to the metric topology  $\mathcal{B}(\mathbb{R})$  induced by the metric  $d_{euclidean}$ , is the usual notion of continuity of Real Analysis.

1. the conditional Wiener measure  $W_t(q', q'')$  is countably-additive on the cylinder sets of  $\mathcal{C}(q', 0; q'', t)$
2. by Kolmogorov Reconstruction Theorem it has a unique extension to the Borel subsets of  $\mathcal{C}(q', 0; q'', t)$  that one defines to be the conditional Wiener measure  $W_t(q', q'')$

Given a potential energy  $V$  bounded from below let us introduce the one-parameter family of operators  $\{T_t^{(V)} := \exp[t(\frac{1}{2}\frac{d^2}{dq^2} - V(q))]\}_{t \in (0, +\infty)}$  and the associated integral kernel:

$$(T_t^{(V)} f)(q') =: \int_{-\infty}^{+\infty} dq'' K_t^{(V)}(q', q'') f(q'') \quad \forall q' \in \mathbb{R}, \forall t \in (0, +\infty) \quad (2.9)$$

The application of the Trotter-Kato formula:

$$\exp(\frac{1}{2}\frac{d^2}{dq^2} - V(q)) = s - \lim_{n \rightarrow +\infty} [\exp(\frac{1}{2n}\frac{d^2}{dq^2}) \exp(-\frac{V}{n})]^n \quad (2.10)$$

allows to derive the following:

### Theorem II.1

*Feynman-Kac formula on the euclidean line:*

$$K_t^{(V)}(q', q'') = \int_{\mathcal{C}(q', 0; q'', t)} dW_t(q', q'') \exp(-\int_0^t ds V[q(s)]) \quad \forall q', q'' \in \mathbb{R}, \forall t \in (0, +\infty) \quad (2.11)$$

### Remark II.1

Though often called simply the brownian motion on  $\mathbb{R}$  the Wiener process has to be more precisely called the brownian motion on the one-dimensional euclidean manifold  $(\mathbb{R}, \delta_{euclidean})$  where  $\delta_{euclidean}$  is the euclidean riemannian metric over  $\mathbb{R}$ .

In fact the definition of the brownian motion involves the choice of a particular  $\sigma$ -algebra over  $\mathbb{R}$  that is the Borel- $\sigma$ -algebra associated to the topology over  $\mathbb{R}$  induced by the metric  $d_{euclidean}$  induced by the norm on  $\mathbb{R}$  induced by the euclidean riemannian-metric  $\delta_{euclidean}$ .

The importance of this conceptual subtlety will appear as soon as we will present the generalization of the Feynman-Kac formula to arbitrary riemannian manifolds.

### III. ANDERSON'S NONSTANDARD CONSTRUCTION OF THE WIENER MEASURE ON THE EUCLIDEAN LINE

The powerful machinery of Loeb measures (reviewed in the section D) has led R.M. Anderson [10] to introduce a wonderful nonstandard construction of the brownian motion (as an infinitesimal random walk) and of the Wiener measure.

Given the time interval  $(0, t), t \in (0, +\infty)$  and an unlimited hypernatural  $n \in {}^*\mathbb{N} - \mathbb{N}$  let us introduce the times:

$$t_k := k\epsilon \quad k \in \{j \in {}^*\mathbb{N} : j \leq n\} \quad (3.1)$$

where:

$$\epsilon := \frac{t}{n} \quad (3.2)$$

Obviously:

$$\epsilon \in \text{hal}(0) \quad (3.3)$$

Let us consider the hyperfinite (time) interval:

$$[0, t]_n := \{t_k, k \in \{j \in {}^*\mathbb{N} : j \leq n\}\} \quad (3.4)$$

and the set:

$$\begin{aligned} \Omega(t; n; q_1, q_2) := \{ \omega : [0, t] \mapsto {}^*\mathbb{R} : \omega(0) = q_1, \omega(t) = q_2, \\ \omega(t_{k+1}) = \omega(t_k) \pm \sqrt{\epsilon} \text{ linearly interpolated between } t_k \text{ and } t_{k+1} \\ \forall k \in \{j \in {}^*\mathbb{N} : j \leq n-1\} \} \end{aligned} \quad (3.5)$$

Let us now introduce the internal probability space  $(\Omega(t; n; q_1, q_2), {}^*\mathcal{P}[\Omega(t; n; q_1, q_2)], W_{t;n}(q_1, q_2))$ , where  $W_{t;n}(q_1, q_2)$  is the counting measure on  $\Omega(t; n; q_1, q_2)$  (see the section D), and the corresponding Loeb probability space  $(\Omega(t; n; q_1, q_2), L({}^*\mathcal{P}[\Omega(t; n; q_1, q_2)]), W_{t;n}(q_1, q_2)_L)$ . Then Robert M. Anderson has proved the following:

#### Theorem III.1

*Anderson's Theorem:*

1. for every Borel set  $B \subseteq \mathcal{C}(q_1, 0; q_2, t)$ :

$$W_{t;n}(q_1, q_2)_L(st^{-1}(B)) = W_t(q_1, q_2)(B) \quad (3.6)$$

where  $W_t(q_1, q_2)$  is the conditional Wiener measure introduced in the previous section

2. the stochastic process  $w : [0, t] \times \Omega(t; n; q_1, q_2) \mapsto \mathbb{R}$  defined by:

$$w_s(\omega) := st[\omega(s)] \quad (3.7)$$

is nothing but the brownian motion over the metric space  $(\mathbb{R}, d_{\text{euclidean}})$  introduced in the previous section.

#### Remark III.1

Anderson's Theorem shows, in the particular case in which the underlying riemannian manifold is the euclidean line, that brownian motion can be defined as an infinitesimal random walk.

We will see that such a characterization can be extended to Hausdorff riemannian manifolds.

#### Remark III.2

Anderson's results have been reformulated also in the language of Internal Set Theory [11], [12], [13].

Since, as we show in section C, Internal Set Theory has great conceptual bugs, such an approach, in our modest opinion, is misleading.

#### IV. NONSTANDARD FEYNMAN-KAC FORMULA ON THE EUCLIDEAN LINE

The idea of avoiding the mathematically meaningless continuum limit of the time-sliced path integrals by taking an hyperfinite time-slicing has been already fruitfully adopted by many authors [14], [15], [16], [17], [18].

Actually, it is sufficient to use the theorem III.1 to substitute Anderson's expression of the Wiener measure into the Feynman-Kac formula on the euclidean real line (i.e. theorem II.1) to obtain the following:

##### Theorem IV.1

*Nonstandard Feynman-Kac formula on the euclidean line:*

$$K_t^{(V)}(q_1, q_2) = \int_{\Omega(t; n; q_1, q_2)} d[W_{t; n}(q_1, q_2)_L(st^{-1})] \exp\left(-\int_0^t ds V[q(s)]\right) \quad \forall q_1, q_2 \in \mathbb{R}, \forall t \in (0, +\infty), \forall n \in {}^*\mathbb{N} - \mathbb{N} \quad (4.1)$$

## V. THE CONDITIONAL WIENER MEASURE AND THE FEYNMAN-KAC FORMULA ON A RIEMANNIAN MANIFOLD

Given a D-dimensional riemannian manifold  $(M, g)$  (with  $g$  given in local coordinates by  $g = g_{\mu\nu} dx^\mu \otimes dx^\nu$ ) let us consider the strongly-continuous contraction semigroup  $\{T_t^{(0)} := \exp(-\frac{1}{2}t\Delta_g)\}_{t \in (0, +\infty)}$  of operators acting on the Hilbert space  $\mathcal{H} := L^2(M, d\mu_g)$  where:

1.  $\Delta_g : \Omega^r(M) \mapsto \Omega^r(M)$ ,  $r \in \mathbb{N}$ :

$$\Delta_g := d d^\dagger + d^\dagger d \quad (5.1)$$

(with  $\Omega^r(M)$  denoting the set of all  $r$ -forms on  $M$ ) is the Laplace-Beltrami operator on  $(M, g)$  [19], [20] of which we consider the restriction to  $\Omega^0(M)$  given in local coordinates by:

$$\Delta_g f = -\frac{1}{\sqrt{|g|}} \partial_\mu [\sqrt{|g|} g^{\mu\nu} \partial_\nu f] \quad (5.2)$$

(where of course  $\partial_\mu := \frac{\partial}{\partial x^\mu}$  while  $g := \det(g_{\mu\nu})$ )

2.  $d\mu_g$  is the invariant measure given in local coordinates by:

$$d\mu_g = \sqrt{|g|} dx^1 \dots dx^D \quad (5.3)$$

Introduced the kernel  $K_t^{(0)}(q_1, q_2)$  of  $\{T_t^{(0)}\}_{t \in (0, +\infty)}$ :

$$(T_t^{(0)} f)(q_1) =: \int_M d\mu_g(q_2) K_t^{(0)}(q_1, q_2) f(q_2) \quad \forall q_1 \in M, \forall t \in (0, +\infty) \quad (5.4)$$

it may be proved [21] that  $K_t^{(0)}(q_1, q_2)$  satisfies the following conditions:

- 1.

$$K_t^{(0)}(q_1, q_2) > 0 \quad \forall q_1, q_2 \in M, \forall t \in (0, +\infty) \quad (5.5)$$

- 2.

$$\int_M d\mu_g(q_2) K_t^{(0)}(q_1, q_2) = 1 \quad \forall q_1 \in M, \forall t \in (0, +\infty) \quad (5.6)$$

- 3.

$$K_{t_1+t_2}^{(0)}(q_1, q_3) = \int_M d\mu_g(q_2) K_{t_1}^{(0)}(q_1, q_2) K_{t_2}^{(0)}(q_2, q_3) \quad \forall q_1, q_3 \in M, \forall t_1, t_2 \in (0, +\infty) \quad (5.7)$$

that guarantee that  $K_t^{(0)}(q_1, q_2)$  is the transition probability kernel of a Markovian stochastic process  $w_t$  that one defines as the brownian motion over  $(M, g)$  <sup>2</sup>.

Given  $q_1, q_2 \in M$  and  $t \in (0, +\infty)$  let us introduce the functional space:

$$\mathcal{C}(q_1, 0; q_2, t) := \{q : [0, t] \mapsto \mathbb{R} \text{ continuous} : q(0) = q_1, q(t) = q_2\} \quad q_1, q_2 \in M, t \in (0, +\infty) \quad (5.8)$$

and the cylinder sets of  $\mathcal{C}(q_1, 0; q_2, t)$ :

$$\Gamma(q_1, 0; \{B_k; t_k\}_{k=1}^{n-1}; q_2, t) := \{q \in \mathcal{C}(q_1, 0; q_2, t) : q(t_k) \in B_k \quad \forall k \in \{j \in \mathbb{N} : 1 \leq j \leq n-1\} \quad 0 < \dots < t_k < t_{k+1} < \dots < t, B_k \in \mathcal{B}_{Borel}(M) \quad \forall k \in \{j \in \mathbb{N} : 1 \leq j \leq n-1\}, n \in \mathbb{N}_+\} \quad (5.9)$$

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<sup>2</sup> Such a definition may be proved to be equivalent to the one obtained through projection from the orthonormal frame bundle [22], [23], [24], [25], [26], [27].



Defined the value of the conditional Wiener measure on cylinder sets as:

$$W[\Gamma(q_1, 0; \{B_k; t_k\}_{k=1}^{n-1}; q_2, t)] := \prod_{k=1}^{n-1} \int_{B_k} d\mu_g(q_k) \prod_{k=1}^n K_{t_{k+1}-t_k}^{(0)}(q_k, q_{k+1}) \quad (5.10)$$

it may be proved that:

1. the conditional Wiener measure  $W_t(q_1, q_2)$  is countably-additive on the cylinder sets of  $\mathcal{C}(q_1, 0; q_2, t)$
2. by Kolmogorov Reconstruction Theorem it has a unique extension to the Borel subsets of  $\mathcal{C}(q_1, 0; q_2, t)$  that one defines to be the conditional Wiener measure  $W_t(q_1, q_2)$

Given a potential energy  $V$  bounded from below, let us introduce the one-parameter family of operators  $\{T_t^{(V)} := \exp[t(-\frac{1}{2}\Delta_g - V(q))]\}_{t \in (0, +\infty)}$  and the associated integral kernel:

$$(T_t^{(V)} f)(q_1) =: \int_M d\mu_g(q_2) K_t^{(V)}(q_1, q_2) f(q_2) \quad \forall q_1 \in M, \forall t \in (0, +\infty) \quad (5.11)$$

The application of the Trotter-Kato formula:

$$\exp(-\frac{1}{2}\Delta_g - V(q)) = s - \lim_{n \rightarrow +\infty} [\exp(-\frac{1}{2n}\Delta_g) \exp(-\frac{V}{n})]^n \quad (5.12)$$

allows to derive the following (see the the section 11.4 "The Feynman Integral and Feynman's Operational Calculus" of [28] and the 7<sup>th</sup> chapter "Symmetries" of [29]):

### Theorem V.1

*Feynman-Kac formula on a riemannian manifold:*

$$K_t^{(V)}(q_1, q_2) = \int_{\mathcal{C}(q_1, 0; q_2, t)} dW_t(q_1, q_2) \exp(-\int_0^t ds V[q(s)]) \quad \forall q_1, q_2 \in M, \forall t \in (0, +\infty) \quad (5.13)$$

### Remark V.1

Theorem V.1 is strongly connected to a long-standing problem debated in the Physics' literature, i.e. the one of quantizing a classical non-relativistic dynamical system describing a particle of unary mass constrained to move on  $(M, g)$  having classical action  $S[q] := \int dt \frac{|\dot{q}|_g^2}{2}$  (see for instance the 24<sup>th</sup> chapter of [3], the 9<sup>th</sup> chapter "Quantization" of [30], the 3<sup>th</sup> chapter "Path integrals in Quantum Mechanics: Generalizations" of [31], the chapters 10 "Short-Time Amplitude in Spaces with Curvature and Torsion" and the chapter 11 "Schrödinger Equation in General Metric-Affine Spaces" of [4], the 15<sup>th</sup> chapter "The Nonrelativistic Particle in a Curved Space" of [32] as well as references therein).

A great amount of the mentioned literature is based on the generalization of Feynman's approach, i.e. on:

1. introducing  $n \in \mathbb{N}_+$  times:

$$t_k := k\epsilon \quad k \in \{j \in \mathbb{N} : 0 \leq j \leq n\} \quad (5.14)$$

where:

$$\epsilon := \frac{t}{n} \quad (5.15)$$

2. using the semi-group property of the euclidean time-evolution operator  $\hat{U}$  in order to write:

$$\langle q'' | \hat{U}_t | q' \rangle = \prod_{k=1}^{n-1} \int_M d\mu_g(q_k) \prod_{k=1}^n \langle q_k | \hat{U}_\epsilon | q_{k-1} \rangle \quad (5.16)$$

where  $q_0 := q'$  and  $q_n := q''$ .

3. obtaining, according to one among different choices in the evaluation-point (claimed to correspond to different operators' orderings), one of the uncountably many possible expressions for  $\langle q_k | \hat{U}_\epsilon | q_{k-1} \rangle$  expressed in local coordinates.

4. performing a mathematically meaningless limit for  $n \rightarrow +\infty$  (and hence  $\epsilon \rightarrow 0$ ) in which the mathematically meaningless object  $\lim_{n \rightarrow +\infty} (\frac{1}{2\pi\epsilon})^{\frac{n}{2}} \prod_{k=1}^{n-1} d\mu_g(q_k)$  is replaced with a mathematically meaningless functional measure  $d\mu_g[q(t)]$

whose mathematical meaning is less (if possible) than Feynman's original approach owing to the factor  $|g|^{\frac{n}{2}}$  obtained from the expressions in local coordinates of  $d\mu_g$  of which each author gets rid of in some way.

Not surprisingly different authors arrive to different conclusions.

Most of them concord on the fact that the quantum hamiltonian should be of the form  $\hat{H} = \frac{1}{2}\Delta_g + cR$  where  $c \in \mathbb{Q}$  and where  $R$  is the scalar curvature of  $(M, g)$ .

As to the value of the number  $c$  the more palatable proposals are  $c = 0$  (as according to Cecile Morette De-Witt),  $c = \frac{1}{6}$  (in conformity with the  $1^{th}$  order term in the asymptotic expansion  $K_t^{(0)}(q, q) \sim \sum_{n=0}^{+\infty} a_n(q)t^n$  [33]),  $c = \frac{1}{12}$  (as according to the first Bryce De-Witt),  $c = \frac{1}{8}$  (as according to the last Bryce De-Witt) and various other alternatives (somewhat related to the fact that, for  $D \geq 2$ ,  $\hat{H}$  is invariant under conformal transformations if and only if  $c = \frac{D-2}{4(D-1)}$ ).

## VI. INFINITESIMAL RANDOM WALKS ON AN HAUSDORFF RIEMANNIAN MANIFOLD

Nonstandard diffusions on manifolds have been studied in [34] using the machinery developed in the 5<sup>th</sup> chapter "Hyperfinitesimal Dirichlet Forms and Markov Processes" of [14].

The treatment therein contained, anyway, doesn't furnish an explicit generalization of Anderson's construction.

The content of section III may be easily generalized to infinitesimal random walks over an Hausdorff riemannian manifold  $(M, g)$ .

Given the time interval  $(0, t), t \in (0, +\infty)$  and an unlimited hypernatural  $n \in {}^*\mathbb{N} - \mathbb{N}$  let us introduce the times:

$$t_k := k\epsilon \quad k \in \{j \in {}^*\mathbb{N} : j \leq n\} \quad (6.1)$$

where:

$$\epsilon := \frac{t}{n} \quad (6.2)$$

Obviously:

$$\epsilon \in \text{hal}(0) \quad (6.3)$$

Let us consider the hyperfinite (time) interval:

$$[0, t]_n := \{t_k \mid k \in \{j \in {}^*\mathbb{N} : j \leq n\}\} \quad (6.4)$$

Given two points  $q_1, q_2 \in M$ :

### Definition VI.1

$$\begin{aligned} \Omega(t; n; q_1, q_2) &:= \{\omega : [0, t] \mapsto {}^*M : \omega(0) = q_1, \omega(t) = q_2, \\ &\quad {}^*d_g[\omega(t_{k+1}), \omega(t_k)] = \pm\sqrt{\epsilon}, \\ &\quad \text{geodetically interpolated between } t_k \text{ and } t_{k+1} \forall k \in \{j \in {}^*\mathbb{N} : 1 \leq j \leq n-1\}\} \end{aligned} \quad (6.5)$$

where the geodetic interpolation between  $\omega(t_k)$  and  $\omega(t_{k+1})$  is the extended shortest geodetic arc connecting these points (whose existence is guaranteed by the Hopf-Rinow Theorem [35] combined with proposition B.10) and  ${}^*d_g[\omega(t_{k+1}), \omega(t_k)]$  is the extended geodesic-distance between  $\omega(t_k)$  and  $\omega(t_{k+1})$ , i.e. the length of such an infinitesimal geodetic arc.

Let us now introduce the internal probability space  $(\Omega(t; n; q_1, q_2), {}^*\mathcal{P}[\Omega(t; n; q_1, q_2)], W_{t;n}(q_1, q_2))$ , where  $W_{t;n}(q_1, q_2)$  is the counting measure on  $\Omega(t; n; q_1, q_2)$ , and the corresponding Loeb probability space  $(\Omega(t; n; q_1, q_2), L({}^*\mathcal{P}[\Omega(t; n; q_1, q_2)]), W_{t;n}(q_1, q_2)_L)$ . Then:

### Theorem VI.1

*Generalized Anderson's Theorem:*

1. for every Borel set  $B \subseteq \mathcal{C}(q_1, 0; q_2, t)$ :

$$W_{t;n}(q_1, q_2)_L(st^{-1}(B)) = W_t(q_1, q_2)(B) \quad (6.6)$$

where  $W_t(q_1, q_2)$  is the conditional Wiener measure introduced in the previous section

2. the stochastic process  $w : [0, t] \times \Omega(t; n; q_1, q_2) \mapsto \mathbb{R}$  defined by:

$$w_s(\omega) := st[\omega(s)] \quad (6.7)$$

is nothing but the brownian motion over  $(M, g)$  introduced in the previous section.

PROOF:

Anderson's proof can be completely formulated in terms of Nonstandard Topology applied to the topological space  $(\mathbb{R}, \mathcal{T}_{\text{natural}})$ .

So, according to proposition B.14, it may be immediately generalized to  $M$ , seen as topological space, provided that it is Hausdorff.

The replacement of the euclidean mathematical objects of Anderson's Theorem on the euclidean manifold  $(\mathbb{R}, \delta)$  with the corresponding riemannian-geometric objects of the riemannian manifold  $(M, g)$  is then straightforward.

■

## VII. NONSTANDARD FEYNMAN-KAC FORMULA ON AN HAUSDORFF RIEMANNIAN MANIFOLD

The approach followed in the section IV may be immediately generalized to the case of an Hausdorff riemannian manifold  $(M, g)$ .

Actually, it is sufficient to use the theorem VI.1 to substitute the generalized Anderson's expression of the Wiener measure into the Feynman-Kac formula on  $(M, g)$  (i.e. the theorem V.1) to obtain the following:

### Theorem VII.1

*Nonstandard Feynman-Kac formula on an Hausdorff riemannian manifold:*

$$K_t^{(V)}(q_1, q_2) = \int_{\Omega(t; n; q_1, q_2)} d[W_{t; n}(q_1, q_2)_L(st^{-1})] \exp\left(-\int_0^t ds V[q(s)]\right) \quad \forall q_1, q_2 \in M, \forall t \in (0, +\infty), \forall n \in {}^*\mathbb{N} - \mathbb{N} \quad (7.1)$$

### VIII. TAKING INTO ACCOUNT NONCOMMUTATING OPERATORS: INFINITESIMAL QUANTUM RANDOM WALKS ON NONCOMMUTATIVE RIEMANNIAN MANIFOLDS

As the link between Quantum Mechanics and brownian motion exhibited by the Feynman-Kac formulas led Edward Nelson to formulate Stochastic Mechanics [36], [37], a reformulation of Quantum Mechanics in terms of classical markovian stochastic processes in which, for instance, the quantization of the nonrelativistic classical dynamical system consisting in a particle of unary mass constrained to the riemannian manifold  $(M, g)$  (and hence described by the classical action functional  $S[q] := \int dt \frac{|\dot{q}|^2}{2}$ ) is performed entirely in terms of stochastic averages with respect to the brownian motion on  $(M, g)$ , one could think to adopt theorem VII.1 as a starting point to formulate a nonstandard version of Stochastic Mechanics in which, for instance, the quantization of the mentioned classical dynamical system would be performed entirely in terms of stochastic averages with respect to the infinitesimal random walk on  $(M, g)$ .

From a mathematical point of view this would allow to formulate in a mathematically and conceptually rigorous way the stochastic variational approach bypassing the problem of the ill-defined nature of the stochastic functional  $S_{stoc}[q] := \int dt E[\frac{|\dot{q}|^2}{2}]$  (owed to the basic property of brownians paths informally expressed as  $(dq)^2 = dt$ <sup>3</sup> and the consequential informal fact that  $\dot{q} := \sqrt{\frac{(dq)^2}{(dt)^2}} = \sqrt{\frac{1}{dt}}$  is ill-defined) and hence also the "resolution" of the problem through the adoption of the controversial renormalization of the stochastic action that, in our modest opinion, is no more than a conjuring trick.

A consistent stochastic variational principle would then be formalizable in terms of the Loeb Measure Theoretic approach to the Malliavin Calculus [38], [39] exposed in the 3<sup>th</sup> chapter "Stochastic Calculus of Variations" of [40].

This would, anyway, contribute to hide a structural limitation of the Feynman-Kac formulas (and consequentially of Stochastic Mechanics) that it is never sufficiently remarked:

though they allow to express the quantum averages of the elements of the commutative Von Neumann subalgebra generated by one operator as expectation values taken with respect to a classical (i.e. commutative) Kolmogorovian probability space, the game breaks up as soon as one takes into accounts noncommutating operators (whose existence is the soul of Quantum Mechanics), as it can be appreciated going higher than the ground floor of the following (see for instance [41], [42], [43], [44] [45], [46], [47], [48], [49], [50]):

#### Theorem VIII.1

*Theorem of the Noncommutative Tower:*

- (ground floor) Noncommutative Topology:

The *category* having as *objects* the *Hausdorff compact topological spaces* and as *morphisms* the *continuous maps* on such spaces is equivalent to the category having as *objects* the *abelian  $C^*$ -algebras* and as *morphisms* the *involutive morphisms* of such spaces.

- (first floor) Noncommutative Probability:

The *category* having as *objects* the *classical probability spaces* and as *morphisms* the *endomorphisms (automorphisms)* of such spaces is equivalent to the *category* having as *objects* the *abelian algebraic probability spaces* and as *morphisms* the *endomorphisms (automorphisms)* of such spaces.

- (second floor) Noncommutative Geometry:

The *category* having as *objects* the *closed finite-dimensional riemannian spin manifolds* and as *morphisms* the *diffeomorphisms of such manifolds* is equivalent to the *category* having as *objects* the *abelian spectral triples* and as *morphisms* the *automorphisms of the involved involutive algebras*.

to infer that as a matter of principle only a limited number of moments of a noncommutative probability space can be reproduced by a commutative probability space (as concretely shown by the difficulties arising as soon as one tries to

---

<sup>3</sup> a consequence of which is the fact that the Wiener measure is supported on paths Hölderian of order  $< \frac{1}{2}$  or, in a more fashionable language, that brownian paths have Hausdorff dimension 2.

give a mathematical foundation to Feynman's operational calculus in terms of integration with respect to the Wiener measure [28])<sup>4</sup>.

Despite Connes' criticisms concerning the claimed nonconstructive nature of the infinitesimals of Nonstandard Analysis (about which we demand to the remark B.1) and his interpretation of the compact operators as the right infinitesimals in the noncommutative framework, it should be possible to use Nonstandard Analysis to reformulate the Sinha-Goswami's definition of a quantum brownian motion on a noncommutative riemannian manifold as an Evans-Hudson dilation of the heat semigroup of the underlying  $C^*$ -algebra (see the 9<sup>th</sup> chapter "Noncommutative Geometry and quantum stochastic process" of [52]) looking at such a quantum brownian motion as an infinitesimal quantum random walks on such a noncommutative riemannian manifold by making the usual ansatz in which a time interval  $[0, t]$ , where  $t \in (0, +\infty)$ , is replaced with an hyperfinite time interval  $[0, t]_n$ ,  $n \in {}^*\mathbb{N} - \mathbb{N}$ .

A preliminary task in this direction would consist in recovering Anderson's construction from the representation of classical brownian motion on  $(\mathbb{R}, d_{euclidean})$  in terms of (what it is natural to call) the hyperfinite family of operators:

$$\{(\hat{a} + \hat{a}^\dagger)(s) \mid s \in [0, t]_n\} \quad (8.1)$$

on the symmetric Fock space  $\Gamma[L^2([0, +\infty), dq) \otimes \mathbb{C}]$  where  $\hat{a}$  and  $\hat{a}^\dagger$  are the usual, respectively, annihilation and creation operators:

$$[\hat{a}(s), \hat{a}^\dagger(s)] = 1 \quad \forall s \in [0, t]_n \quad (8.2)$$

and where as usual  $[0, t]_n$ , for  $t \in (0, +\infty)$  and  $n \in {}^*\mathbb{N} - \mathbb{N}$ , is the hyperfinite (time) interval:

$$[0, t]_n := \{k \cdot \epsilon, k \in \{j \in {}^*\mathbb{N} : j \leq n\}\} \quad (8.3)$$

$$\epsilon := \frac{t}{n} \in hal(0) \quad (8.4)$$

---

<sup>4</sup> One could, at this point, object that to make such an inference it is enough to go to the first floor and there is no necessity to raise to the second floor. This leads us directly to the issue about the hierarchy existing between Probability and Geometry. The  $\sigma$ -algebra of the measurable spaces most used in Theoretical and Mathematical Physics is the Borel- $\sigma$  algebra induced by some topology. In many cases such a topology is the (Hausdorff) metric one induces by a metric depending from an underlying geometric structure; for example it could be the geodesic distance arising from a riemannian manifold's structure. As to the set of all the probability measures on a measurable space, Information Geometry [51] taught us its underlying geometric structure. Hence, despite the appearances, in many cases we see that Geometry is hierarchically prior than Measure Theory (and hence Probability Theory). The same situation occurs in the noncommutative context where in many cases the Von Neumann algebra  $A$  of a noncommutative probability space  $(A, \omega)$  has a noncommutative geometric underlying structure (unfortunately appreciable only after having digested  $C^*$ -modules, cyclic cohomology, crossed products and whatsoever).

## IX. ACKNOWLEDGEMENTS

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## APPENDIX A: THE ORTHODOX ZFC+CH SET-THEORETIC FOUNDATION OF MATHEMATICS

It is nowadays common opinion in the Scientific Community that:

1. The foundations of Mathematics lies on Set Theory
2. Set Theory is axiomatized by the formal system ZFC, i.e. the Zermelo-Fraenkel formal system augmented with the Axiom of Choice

In this section we will strongly defend such an orthodox viewpoint presenting ZFC in detail [53].

Before the birth of ZFC Set Theory was studied by a naive approach (that we will denote as Naive Set Theory) based on the following:

### AXIOM A.1

*Axiom of Frege's-Comprehension:*

If  $p$  is a unary predicate then there exists a set  $\mathbb{U}_p := \{x : p(x)\}$  of all elements having the property  $p$

Naive Set Theory was proved to be inconsistent by Russell who showed that the application of the axiom A.1 to the unary predicate  $p_{\text{Russell}}(x) := x \notin x$  leads to the contradiction:

$$S_{p_{\text{Russell}}} \in S_{p_{\text{Russell}}} \Leftrightarrow S_{p_{\text{Russell}}} \notin S_{p_{\text{Russell}}} \quad (\text{A1})$$

The conceptual earthquake caused by Russell's remark led the Scientific Community to think that a more refined axiomatization of Set Theory was needed.

The result of the deep work of many mathematicians was the formulation of the following:

### Definition A.1

*Formal System of Zermelo-Fraenkel (ZF):*

#### 1. AXIOM A.2

*Axiom of Existence of the empty set:*

There exist the empty set:

$$\exists \emptyset : \forall x \neg(x \in \emptyset) \quad (\text{A2})$$

#### 2. AXIOM A.3

*Axiom of Extensionality:*

If  $x$  and  $y$  have the same elements, then  $x$  is equal to  $y$ :

$$\forall x \forall y [\forall z (z \in x \Leftrightarrow z \in y) \Rightarrow x = y] \quad (\text{A3})$$

#### 3. AXIOM A.4

*Axiom of Comprehension:*

For every formula  $\phi(s, t)$  with free variables  $s$  and  $t$ , for every  $x$ , and for every parameter  $p$  there exists a set  $y := \{u \in x : \phi(u, p)\}$ :

$$\forall x \forall p \exists y : [\forall u (u \in y \Leftrightarrow (u \in x \wedge \phi(u, p)))] \quad (\text{A4})$$

#### 4. AXIOM A.5

*Axiom of Pairing:*

For any  $a$  and  $b$  there exists a set  $x$  that contains  $a$  and  $b$ :

$$\forall a \forall b \exists x : (a \in x \wedge b \in x) \quad (\text{A5})$$



### 5. AXIOM A.6

*Axiom of Union:*

For every family  $\mathcal{F}$  there exists a set  $U$  containing the union  $\cup \mathcal{F}$  of all the elements of  $\mathcal{F}$ :

$$\forall \mathcal{F} \exists U : \forall Y \forall x [x \in Y \wedge Y \in \mathcal{F} \Rightarrow x \in U] \quad (\text{A6})$$

### 6. AXIOM A.7

*Axiom of the Power Set:*

For every set  $X$  there exists a set  $P$  containing the set  $\mathcal{P}(X)$  (called the power set of  $X$ ) of all subsets of  $X$ :

$$\forall X \exists P : \forall z [z \subset X \Rightarrow z \in P] \quad (\text{A7})$$

### 7. AXIOM A.8

*Axiom of Infinity:*

there exists an infinite set (of some special form):

$$\exists x : [\forall z (z = \emptyset \Rightarrow z \in x) \wedge \forall y \in x \forall z (z = \text{Suc}(y) \Rightarrow z \in x)] \quad (\text{A8})$$

where  $\text{Suc}(x) := x \cup \{x\}$  is called the successor of  $x$ .

### 8. AXIOM A.9

*Axiom of Replacement:*

For every formula  $\phi(s, t, U, w)$  with free variables  $s, t, U$  and  $w$ , for every set  $A$  and for every parameter  $p$ , if  $\phi(s, t, A, p)$  defines a function  $F$  on  $A$  by  $F(x) = y \Leftrightarrow \phi(x, y, A, p)$ , then there exists a set  $Y$  containing the range  $F[A] := \{F(x) : x \in A\}$  of the function  $F$ :

$$\forall A \forall p [\forall x \in A \exists! y : \phi(x, y, A, p) \Rightarrow \exists Y : \forall x \in A \exists y \in Y : \phi(x, y, A, p)] \quad (\text{A9})$$

### 9. AXIOM A.10

*Axiom of Foundation:*

Every non-empty set has an  $\in$ -minimal element:

$$\forall x [\exists y : (y \in x) \Rightarrow \exists y : (y \in x \wedge \neg \exists z : (z \in x \wedge z \in y))] \quad (\text{A10})$$

### Remark A.1

Let us remark that, to shorten the notation, we have used in the definition A.1 the symbol  $\cup$  of union and the symbol  $\subset$  of inclusion though the only (undefined) unary predicate contained in the definition A.1 is the predicate  $\in$  of memberships.

Actually all the other set-theoretic connectives are defined in terms of it.

So, given two sets  $S_1$  and  $S_2$ :

### Definition A.2

*union of  $S_1$  and  $S_2$ :*

$$S_1 \cup S_2 := \{x : x \in S_1 \vee x \in S_2\} \quad (\text{A11})$$

### Definition A.3

*intersection of  $S_1$  and  $S_2$*

$$S_1 \cap S_2 := \{x : x \in S_1 \wedge x \in S_2\} \quad (\text{A12})$$

**Definition A.4**

$S_1$  is a subset of  $S_2$ :

$$S_1 \subseteq S_2 := x \in S_2 \quad \forall x \in S_1 \quad (\text{A13})$$

**Definition A.5**

$S_1$  is a proper subset of  $S_2$ :

$$S_1 \subset S_2 := S_1 \subseteq S_2 \wedge S_1 \neq S_2 \quad (\text{A14})$$

**Remark A.2**

Let us remark that the Axiom of Foundation (i.e. the axiom A.10) is nonconstructive: it assures us that given a non-empty set  $S$  there exist an element  $m(S)$  of  $S$  that is  $\in$ -minimal, but it doesn't gives an algorithm that, receiving  $S$  as input, gives  $m(S)$  as output.

Let us assume the formal system ZF.

Given arbitrary  $a$  and  $b$ :

**Definition A.6**

ordered pair of  $a$  and  $b$ :

$$\langle a, b \rangle := \{\{a\}, \{a, b\}\} \quad (\text{A15})$$

Given two sets  $S_1$  and  $S_2$ :

**Definition A.7**

cartesian product of  $S_1$  and  $S_2$ :

$$S_1 \times S_2 := \{z \in \mathcal{P}(\mathcal{P}(S_1 \cup S_2)) : \exists x \in S_1 \exists y \in S_2 : (z = \langle x, y \rangle)\} \quad (\text{A16})$$

**Definition A.8**

binary relation between  $S_1$  and  $S_2$ :

$$R \in \mathcal{P}(S_1 \times S_2) \quad (\text{A17})$$

Given a binary relation  $R$  between  $S_1$  and  $S_2$  let introduce the notation:

**Definition A.9**

$$xRy := \langle x, y \rangle \in R \quad (\text{A18})$$

**Definition A.10**

$R$  is a map with domain  $S_1$  and codomain  $S_2$ :

$$\forall x \in S_1, \exists! R(x) \in S_2 : xRR(x) \quad (\text{A19})$$

A map  $f$  with domain  $S_1$  and codomain  $S_2$  (briefly a map from  $S_1$  to  $S_2$ ) is denoted as  $f : S_1 \mapsto S_2$ .

**Definition A.11**

set of all maps with domain  $S_1$  and codomain  $S_2$

$$S_2^{S_1} := \{f : S_1 \mapsto S_2\} \quad (\text{A20})$$

Given a binary relation  $R$  on a set  $S$ :

**Definition A.12**

*R is a preordering:*

1. it is reflexive:

$$xRx \quad \forall x \in S \quad (\text{A21})$$

2. it is transitive:

$$[(x_1Rx_2 \wedge x_2Rx_3) \Rightarrow x_1Rx_3] \quad \forall x_1, x_2, x_3 \in S \quad (\text{A22})$$

**Definition A.13**

*R is an equivalence relation:*

1. it is a preordering
2. it is symmetric:

$$(x_1Rx_2 \Rightarrow x_2Rx_1) \quad \forall x_1, x_2 \in S \quad (\text{A23})$$

**Definition A.14**

*R is a partial ordering over S:*

1. it is a preordering
2. it is antisymmetric:

$$[(x_1Rx_2 \wedge x_2Rx_1) \Rightarrow x_1 = x_2] \quad \forall x_1, x_2 \in S \quad (\text{A24})$$

**Definition A.15**

*R is a total ordering over S:*

1. it is a partial ordering over S
- 2.

$$(x_1Rx_2 \vee x_2Rx_1) \quad \forall x_1, x_2 \in S \quad (\text{A25})$$

**Definition A.16**

*R is well-founded:*

$$[Y \neq \emptyset \Rightarrow \exists m \in Y : (\nexists y \in Y : yRm)] \quad \forall Y \subset S \quad (\text{A26})$$

**Definition A.17**

*R is a well-ordering over S:*

1. it is a total ordering over S
2. it is well-founded

We have now all the ingredients required to introduce in a compact way the following:

**Definition A.18**

*formal system of Zermelo-Fraenkel augmented with the Axiom of Choice (ZFC):*  
the formal system ZF augmented with the following:

**AXIOM A.11**

*Axiom of Choice:*

$$\forall S \neq \emptyset \exists f \in (\cup_{B \in \mathcal{P}(S)} B)^{\mathcal{P}(S)} : f(A) \in A \quad \forall A \in \mathcal{P}(S) : A \neq \emptyset \quad (\text{A27})$$

**Remark A.3**

Let us remark that the Axiom of Choice (i.e. axiom A.11) is nonconstructive: it assures us that given a non-empty set  $S$  there exists a function  $f_S$  that maps each non-empty subset into an element of it but it doesn't give us an algorithm that, receiving as input the set  $S$ , gives as output the map  $f_S$ .

**Remark A.4**

The nonconstructive nature of the Axiom of Choice (i.e. axiom A.11) has led many mathematicians and physicists to look with suspicion at the results depending on it (such as the Hahn-Banach Theorem in Functional Analysis).

Gödel has proved that the Axiom of Choice is consistent relative to ZF, i.e. that if ZF is consistent then ZF augmented with the Axiom of Choice is consistent too.

However Paul Cohen has proved that also the negation of the Axiom of Choice is consistent relative to ZF.

Horst Herrlich has recently published a very interesting book [54] comparing the disasters occurring avoiding the Axiom of Choice with the ones occurring assuming the Axiom of Choice.

As to Mathematical-Physics it should be remarked that while the disasters (such as the famous Banach-Tarski Paradox according to which, within ZFC, any two bounded subsets  $A$  and  $B$  of  $\mathbb{R}^3$ , each one containing some ball, are equidecomposable) caused by the Axiom of Choice are easily exorcizable (as to the Banach-Tarski Paradox, for instance, this is automatically done by the non Lebesgue-measurability of the pieces of the paradoxical decompositions) the disasters caused by the absence of the Axiom of Choice (such as vector spaces having no bases or having bases of different cardinalities) drastically compromise any mathematical foundation of Quantum Mechanics.

Let us assume the formal system ZFC of which we want here to show some key features:

**Theorem A.1**

*Zermelo's Theorem:*

$$\forall S \neq \emptyset \exists R : R \text{ is a well-ordering over } S \quad (\text{A28})$$

**Remark A.5**

Let us remark that the Axiom of Comprehension (i.e. axiom A.4) is different from the Axiom of Frege's-Comprehension (i.e. axiom A.1) since given a unary predicate  $p$ :

1. given a set  $S$  it allows to define the set  $S_p := \{x \in S : p(x)\}$
2. since the undefined object "the proper class  $\mathbb{U}$  of all sets" is not a set in ZFC it doesn't exist within ZFC a set  $\mathbb{U}_p := \{x \in \mathbb{U} : p(x)\}$

and hence Russell's paradox doesn't occur within ZFC.

**Remark A.6**

The axiom A.10 (that can be compactly stated as the condition that the binary relation  $\in$  is well-founded on every non-empty set) implies that:

**Theorem A.2**

1.

$$S \notin S \quad \forall S \quad (\text{A29})$$

2.

$$\nexists \{S_n\}_{n \in \mathbb{N}} : (S_{n+1} \in S_n \quad \forall n \in \mathbb{N}) \quad (\text{A30})$$

**Remark A.7**

The Axiom of Existence of the Empty Set (i.e. axiom A.2) together with the Axiom of Infinity (i.e. axiom A.8) allows to prove the following:

**Theorem A.3**

*Existence and unicity of the set of all natural numbers:*

There exists exactly one set  $\mathbb{N}$  such that:

1.

$$\emptyset \in \mathbb{N} \quad (\text{A31})$$

2.

$$Suc(x) \in \mathbb{N} \quad \forall x \in \mathbb{N} \quad (\text{A32})$$

where, as in the Axiom of Infinity (i.e axiom A.8),  $Suc(x) := x \cup \{x\}$ .

3. if  $K$  is any set that satisfies eq. A31 and eq. A32 then  $\mathbb{N} \subset K$ .

PROOF:

By the Axiom of Infinity (i.e. axiom A.8) there exists at least one set  $X$  satisfying eq. A31 and eq. A32.

Let:

$$\mathcal{F} := \{Y \in \mathcal{P}(X) : \emptyset \in Y \wedge (Suc(x) \in Y \quad \forall x \in Y)\} \quad (\text{A33})$$

$$\mathbb{N} := \bigcap_{Y \in \mathcal{F}} Y \quad (\text{A34})$$

It is easy to see that the intersection of any nonempty family of sets satisfying eq. A31 and eq. A32 still satisfies eq. A31 and eq. A32.

Let  $K$  be a set that satisfies eq. A31 and eq. A32; then:

$$X \cap K \in \mathcal{F} \quad (\text{A35})$$

and:

$$\mathbb{N} := \bigcap_{Y \in \mathcal{F}} Y \subset X \cap K \subset K \quad (\text{A36})$$

■

**Remark A.8**

Let us remark that the set  $\mathbb{N}$  whose existence and unicity is stated by theorem A.3 is recursive [55], [56].

Actually every element of its can be concretely computed through the following Mathematica [57] expressions:



[illegible]

10 =

[illegible]

and so on.

### Definition A.19

*sum of natural numbers:*

$$+ \in \mathbb{N}^{\mathbb{N}^2} :$$

$$n + 0 := n \tag{A37}$$

$$n+1 := Suc(n) \tag{A38}$$

$$n + (m + 1) := (n + m) + 1 \quad (\text{A39})$$

### Definition A.20

*product of natural numbers:*

 $\cdot \in \mathbb{N}^{\mathbb{N}^2}:$ 

$$n \cdot 0 := 0 \tag{A40}$$

$$n \cdot (m + 1) := (n \cdot m) + n \quad (\text{A41})$$

**Definition A.21**

*exponentiation of natural numbers:*

$\cdot \in \mathbb{N}^{\mathbb{N}^2}$ :

$$n^0 := 1 \quad (\text{A42})$$

$$n^{m+1} := n^m \cdot n \quad (\text{A43})$$

**Definition A.22**

*ordering of natural numbers:*

$<, \leq \in \mathcal{P}(\mathbb{N}^2)$ :

$$m < n := m \in n \quad (\text{A44})$$

$$m \leq n := m \subset n \quad (\text{A45})$$

**Definition A.23**

*set of all integer numbers:*

$$\mathbb{Z} := \frac{\mathbb{N}^2}{\sim_{\mathbb{Z}}} \quad (\text{A46})$$

where  $\sim_{\mathbb{Z}}$  is the following equivalence relation over  $\mathbb{N}^2$ :

$$\langle n_1, m_1 \rangle \sim_{\mathbb{Z}} \langle n_2, m_2 \rangle := n_1 + m_2 = n_2 + m_1 \quad (\text{A47})$$

**Definition A.24**

*sum of integer numbers:*

$+$   $\in \mathbb{Z}^{\mathbb{Z}^2}$ :

$$[\langle n_1, m_1 \rangle] + [\langle n_2, m_2 \rangle] := [\langle n_1 + n_2, m_1 + m_2 \rangle] \quad (\text{A48})$$

**Definition A.25**

*subtraction of integer numbers:*

$-$   $\in \mathbb{Z}^{\mathbb{Z}^2}$ :

$$[\langle n_1, m_1 \rangle] - [\langle n_2, m_2 \rangle] := [\langle n_1, m_1 \rangle] + [\langle m_2, n_2 \rangle] \quad (\text{A49})$$

**Definition A.26**

*product of integer numbers:*

$\cdot$   $\in \mathbb{Z}^{\mathbb{Z}^2}$ :

$$[\langle n_1, m_1 \rangle] \cdot [\langle n_2, m_2 \rangle] := [\langle n_1 \cdot n_2 + m_1 \cdot m_2, n_1 \cdot m_2 + m_1 \cdot n_2 \rangle] \quad (\text{A50})$$

**Definition A.27**

*ordering of integer numbers:*

$\leq \in \mathcal{P}(\mathbb{Z}^2)$ :

$$[\langle n_1, m_1 \rangle] \leq [\langle n_2, m_2 \rangle] := n_1 + m_2 \leq n_2 + m_1 \quad (\text{A51})$$



**Definition A.28**

set of all rational numbers

$$\mathbb{Q} := \{[< a, b >] : a, b \in \mathbb{Z} \wedge b \neq 0\} \quad (\text{A52})$$

where the equivalence classes are taken with respect to the the following equivalence relation  $\sim_{\mathbb{Q}}$  over  $\mathbb{Z}^2$ :

$$< a_1, b_1 > \sim_{\mathbb{Q}} < a_2, b_2 > := [(a_1 \cdot b_2 = a_2 \cdot b_1) \wedge (b_1 \neq 0 \neq b_2)] \vee [b_1 = b_2 = 0] \quad (\text{A53})$$

**Definition A.29**

sum of rational numbers:

$+ \in \mathbb{Q}^{\mathbb{Q}^2}$ :

$$[< a_1, b_1 >] + [< a_2, b_2 >] := [< a_1 \cdot b_2 + a_2 \cdot b_1, b_1 \cdot b_2 >] \quad (\text{A54})$$

**Definition A.30**

product of rational numbers:

$\cdot \in \mathbb{Q}^{\mathbb{Q}^2}$ :

$$[< a_1, b_1 >] \cdot [< a_2, b_2 >] := [< a_1 \cdot a_2, b_1 \cdot b_2 >] \quad (\text{A55})$$

**Definition A.31**

ordering of rational numbers:

$\leq \in \mathcal{P}(\mathbb{Q}^2)$ :

$$[< a_1, b_1 >] \leq [< a_2, b_2 >] := b_1 \geq 0 \wedge b_2 \geq 0 \wedge a_1 \cdot b_2 \leq a_2 \cdot b_1 \quad (\text{A56})$$

**Definition A.32**

unary real interval:

$$[0, 1] := \frac{\{0, 1\}^{\mathbb{N}}}{\sim_{\mathbb{R}}} \quad (\text{A57})$$

where  $\sim_{\mathbb{R}}$  is the following equivalence relation over  $\{0, 1\}^{\mathbb{N}}$ :

$$\begin{aligned} \{a_n\}_{n \in \mathbb{N}} \sim_{\mathbb{R}} \{b_n\}_{n \in \mathbb{N}} := \\ (\{a_n\}_{n \in \mathbb{N}} = \{b_n\}_{n \in \mathbb{N}}) \vee (\exists n \in \mathbb{N} : \forall k \in \mathbb{N} [(k < n \Rightarrow a_k = b_k) \wedge (a_n = 1 \wedge b_n = 0) \wedge (k > n \Rightarrow a_k = 0 \wedge b_k = 1)]) \end{aligned} \quad (\text{A58})$$

**Definition A.33**

ordering on  $[0, 1]$ :

$\leq \in \mathcal{P}([0, 1]^2)$ :

$$[\{a_n\}_{n \in \mathbb{N}}] \leq [\{b_n\}_{n \in \mathbb{N}}] := [\{a_n\}_{n \in \mathbb{N}}] = [\{b_n\}_{n \in \mathbb{N}}] \vee [\exists n \in \mathbb{N} : a_n < b_n \wedge (a_k = b_k \ \forall k \in n)] \quad (\text{A59})$$

**Definition A.34**

set of all real numbers:

$$\mathbb{R} := \mathbb{Z} \times [0, 1] \quad (\text{A60})$$

where:

$$[0, 1] := [0, 1] - \{1\} \quad (\text{A61})$$

**Definition A.35**

ordering on  $\mathbb{R}$ :  
 $\leq \in \mathcal{P}(\mathbb{R}^2)$ :

$$< k, r > \leq < l, s > := k < l \vee (k = l \wedge r \leq s) \quad (\text{A62})$$

Let us now briefly review how the theory of ordinal and cardinal numbers is introduced within ZFC.

**Definition A.36**

*partially ordered set:*

a couple  $(S, \leq)$  such that:

1.  $S$  is a set
2.  $\leq$  is a partial ordering over  $S$

**Definition A.37**

*totally ordered set:*

a couple  $(S, \leq)$  such that:

1.  $S$  is a set
2.  $\leq$  is a total ordering over  $S$

**Definition A.38**

*well-ordered set:*

a couple  $(S, \leq)$  such that:

1.  $S$  is a set
2.  $\leq$  is a well-ordering over  $S$

Given two partially ordered sets  $(S_1, \leq_1)$  and  $(S_2, \leq_2)$ :

**Definition A.39**

$(S_1, \leq_1)$  and  $(S_2, \leq_2)$  have the same order-type:

$$(S_1, \leq_1) \sim_{ord} (S_2, \leq_2) := \exists f \in S_2^{S_1} \text{ bijective} : [(x \leq_1 y \Leftrightarrow f(x) \leq_2 f(y)) \forall x, y \in S_1] \quad (\text{A63})$$

Given a set  $\alpha$ :

**Definition A.40**

$\alpha$  is an ordinal number:

- 1.

$$\beta \in \alpha \Rightarrow \beta \subset \alpha \quad (\text{A64})$$

- 2.

$$(\beta = \gamma \vee \beta \in \gamma \vee \gamma \in \beta) \quad \forall \beta, \gamma \in \alpha \quad (\text{A65})$$

- 3.

$$\emptyset \neq \beta \subset \alpha \Rightarrow \exists \gamma \in \beta : \gamma \cap \beta = \emptyset \quad (\text{A66})$$

Let us furnish, for completeness, also the following [58], [59], [60]:

**Definition A.41**

$\alpha$  is a surreal number:

$$\exists \beta \text{ ordinal number} : \alpha \in \{0, 1\}^\beta \quad (\text{A67})$$

**Theorem A.4**

1.

$$n \text{ is an ordinal number} \quad \forall n \in \mathbb{N} \quad (\text{A68})$$

2.

$$\mathbb{N} \text{ is an ordinal number} \quad (\text{A69})$$

3.

$$\alpha + 1 := \text{Suc}(\alpha) \text{ is an ordinal number} \quad \forall \alpha \text{ ordinal number} \quad (\text{A70})$$

4.

$$(\alpha, \leq) \text{ is a well-ordered set} \quad \forall \alpha \text{ ordinal number} \quad (\text{A71})$$

where  $\leq := \subset$ .

5.

$$[(\alpha, \leq) \sim_{ord} (\beta, \leq) \Rightarrow \alpha = \beta] \quad \forall \alpha, \beta \text{ ordinal numbers} \quad (\text{A72})$$

6.

$$(\alpha = \beta \vee \alpha < \beta \vee \beta < \alpha) \quad \forall \alpha, \beta \text{ ordinal numbers} \quad (\text{A73})$$

where  $< := \in$ .

Given an ordinal number  $\alpha$ :

**Definition A.42**

$\alpha$  is an ordinal successor:

$$\exists \beta \text{ ordinal number} : \alpha = \text{Suc}(\beta) \quad (\text{A74})$$

**Definition A.43**

$\alpha$  is a limit ordinal:

$$\nexists \beta \text{ ordinal number} : \alpha = \text{Suc}(\beta) \quad (\text{A75})$$

**Example A.1**

Every  $n \in \mathbb{N}$  as well as  $\mathbb{N} + 1, \dots, \mathbb{N} + n$  are ordinal successors.

$\mathbb{N}$ , instead, is a limit ordinal.

**Theorem A.5**

*ordinal numbers as demarcators of order-type:*

HP:

$(W, \leq)$  well-ordered set

TH:

$$\exists! \alpha \text{ ordinal number} : (W, \leq) \sim_{ord} (\alpha, \subset) \quad (\text{A76})$$

Given two sets  $S_1$  and  $S_2$ :

**Definition A.44**

*$S_1$  and  $S_2$  have the same cardinality:*

$$S_1 \sim_{card} S_2 := \exists f \in S_2^{S_1} \text{ bijective} \quad (\text{A77})$$

Given a set  $S$ :

**Definition A.45**

*cardinality of  $S$ :*

$$|S| := \min\{\alpha : \alpha \text{ is an ordinal number} \wedge S \sim_{card} \alpha\} \quad (\text{A78})$$

The name of the definition A.44 is justified by the fact that given two sets  $S_1$  and  $S_2$ :

**Theorem A.6**

$$S_1 \sim_{card} S_2 \Leftrightarrow |S_1| = |S_2| \quad (\text{A79})$$

Given a set  $S$ :

**Definition A.46**

*$S$  is finite:*

$$|S| \in \mathbb{N} \quad (\text{A80})$$

**Definition A.47**

*$S$  is infinite:*

$$|S| \notin \mathbb{N} \quad (\text{A81})$$

**Definition A.48**

*$S$  is countable:*

$$|S| = \aleph_0 \quad (\text{A82})$$

**Definition A.49**

*$S$  is uncountable:*

$$|S| > \aleph_0 \quad (\text{A83})$$

**Definition A.50**

*S has the cardinality of the continuum:*

$$|S| = |\mathbb{R}| \quad (\text{A84})$$

Let us introduce the following:

**Definition A.51**

*formal system ZFC+CH:*

the formal system ZFC augmented with the following:

**AXIOM A.12**

*Continuum Hypothesis (CH):*

$$|\mathbb{R}| = |\mathcal{P}(\mathbb{N})| \quad (\text{A85})$$

**Remark A.9**

The Continuum Hypothesis (i.e. axiom A.12) is rather intuitive since it states that there don't exist "intermediate degrees" of infinity between the discrete (having the cardinality of  $\mathbb{N}$ ) and the continuum (having the cardinality of  $\mathbb{R}$ ).

The reason why one has to add it as a new axiom is that Paul Cohen has proved that it is undecidable (i.e. it can be neither proved nor disproved) within ZFC.

Gödel has, anyway, proved that ZFC+CH is consistent relative to ZFC, i.e. that if ZFC is consistent it follows that ZFC+CH is consistent too.

We will assume that the formal system giving foundation to Mathematics is ZFC+CH that, in particular, we will assume in this paper.

**Remark A.10**

It may be appropriate to conclude this section with a peroration in favor of the ZFC-orthodoxy.

The foundation of Mathematics given by ZFC has passed the test of nearly a century with excellent results.

Though, according to Gödel's Second Theorem [56], we cannot prove the consistency of ZFC from within ZFC itself, no inconsistency in it has been found.

So why to give up to it ?

Yes, it is true that within ZFC one cannot consider proper classes and extend by them the Comprehension Scheme (such as in the Von Neumann Bernays-Gödel augmented with the Axiom of Choice (VNBGC) formal system [61], but is this something really useful considering that provability in ZFC implies provability in VNBGC and Schoenfeld's Theorem stating that a sentence involving only set variables provable in VNBGC is provable also in ZF ?

Or hasn't Peter Aczel's Theory of Hypersets (see for instance the Appendix B "Axioms and Universes" of [63]) obtained giving up the Axiom of Foundation by allowing the membership relation  $\in$  to be not well-founded (well-foundedness condition that can be rephrased as the assumption that decorations are defined only for trees) in the more radical way (i.e. assuming, through the Axiom of Antifoundation, that every graph can be decorated) simply put us again on the edge of Russell's abyss allowing the case in which  $x \in x$  for exoterical intrinsically non-recursive mathematical objects such as the hyperset  $x := Suc(x)$ , implemented by the Mathematica expression:

```
$RecursionLimit=Infinity;
```

```
x := { x , {x} }
```

obviously non-halting ?

Or has the alternative topos-theoretic foundation of Mathematics given by the first-order theory (WPT) of well-pointed topoi presented some concrete advantages balancing the discouragement of having to handle intuitionistic logic and the fact that WPT hasn't the full strength of ZFC as it is shown by the fact that it is equiconsistent only with the weaker formal system RZC of Restricted Zermelo set theory augmented with the Axiom of Choice in which constraints are posed on the adoption of quantifiers in the Comprehension Axiom (see the 10<sup>th</sup> section "Topos Theoretic and Set Theoretic Foundations" of the 6<sup>th</sup> chapter "Topoi and Logic" of [62]) ?

Experience have taught us to defend ZFC's orthodoxy.

As to extensions of ZFC, while the passage from ZFC to ZFC+CH has not bad consequences, there exist pernicious extensions of ZFC with catastrophic consequences, as we will show later.

## APPENDIX B: NONSTANDARD ANALYSIS REMAINING INSIDE ZFC+CH

In this section we present a brief review of the orthodox foundations of Nonstandard Analysis given remaining within the formal system ZFC+CH of definition A.51 [64], [65], [66].

Given a set  $S \neq \emptyset$ :

### Definition B.1

*filter on S:*

$$\mathcal{F} \subseteq \mathcal{P}(S) : (A \cap B \in \mathcal{F} \ \forall A, B \in \mathcal{F}) \wedge [(A \in \mathcal{F}) \wedge (A \subseteq B \subseteq S \Rightarrow B \in \mathcal{F})] \quad (\text{B1})$$

### Definition B.2

*ultrafilter on S:*

a filter  $\mathcal{F}$  on S such that:

$$\mathcal{F} \neq \mathcal{P}(S) \wedge (A \in \mathcal{F} \vee S - A \in \mathcal{F} \ \forall A \in \mathcal{P}(S)) \quad (\text{B2})$$

Given  $B \in \mathcal{P}(S) : B \neq \emptyset$ :

### Definition B.3

*principal filter generated by B:*

$$\mathcal{F}^B := \{A \in \mathcal{P}(S) : A \supseteq B\} \quad (\text{B3})$$

A consequence of the Axiom of Choice (i.e axiom A.11) is the following [65]:

### Proposition B.1

$$|S| \geq |\mathbb{N}| \Rightarrow \exists \mathcal{F} \text{ nonprincipal ultrafilter on S} \quad (\text{B4})$$

Given  $\bar{r} = \{r_n\}_{n \in \mathbb{N}}, \bar{s} = \{s_n\}_{n \in \mathbb{N}} \in \mathbb{R}^{\mathbb{N}}$ :

### Definition B.4

$$\bar{r} \oplus \bar{s} := \{r_n + s_n\}_{n \in \mathbb{N}} \quad (\text{B5})$$

$$\bar{r} \odot \bar{s} := \{r_n \cdot s_n\}_{n \in \mathbb{N}} \quad (\text{B6})$$

Given  $x \in \mathbb{R}$ :

**Definition B.5**

$$x^{\mathbb{N}} := \text{the only element of } \{x\}^{\mathbb{N}} \quad (\text{B7})$$

Let us now introduce the following:

**Definition B.6**

$$NPU(\mathbb{N}) := \{\mathcal{F} \text{ nonprincipal ultrafilter on } \mathbb{N}\} \quad (\text{B8})$$

By Proposition B.1 it follows that:

**Proposition B.2**

$$NPU(\mathbb{N}) \neq \emptyset \quad (\text{B9})$$

Given  $\mathcal{F} \in NPU(\mathbb{N})$  and  $\bar{r} = \{r_n\}_{n \in \mathbb{N}}, \bar{s} = \{s_n\}_{n \in \mathbb{N}} \in \mathbb{R}^{\mathbb{N}}$ :

**Definition B.7**

$\bar{r}$  and  $\bar{s}$  are equal  $\mathcal{F}$ -almost everywhere:

$$\bar{r} \sim_{\mathcal{F}} \bar{s} := \{n \in \mathbb{N} : r_n = s_n\} \in \mathcal{F} \quad (\text{B10})$$

It may be proved that [65]:

**Proposition B.3**

$\sim_{\mathcal{F}}$  is an equivalence relation over  $\mathbb{R}^{\mathbb{N}}$

Let us finally introduce the following:

**Definition B.8**

$${}^*\mathbb{R}_{\mathcal{F}} = \frac{\mathbb{R}^{\mathbb{N}}}{\sim_{\mathcal{F}}} \quad (\text{B11})$$

Given  $\bar{r} = \{r_n\}_{n \in \mathbb{N}}, \bar{s} = \{s_n\}_{n \in \mathbb{N}} \in \mathbb{R}^{\mathbb{N}}$ :

**Definition B.9**

1.

$$[\bar{r}]_{\mathcal{F}} + [\bar{s}]_{\mathcal{F}} := [\bar{r} \oplus \bar{s}]_{\mathcal{F}} \quad (\text{B12})$$

2.

$$[\bar{r}]_{\mathcal{F}} \cdot [\bar{s}]_{\mathcal{F}} := [\bar{r} \odot \bar{s}]_{\mathcal{F}} \quad (\text{B13})$$

3.

$$[\bar{r}]_{\mathcal{F}} \leq [\bar{s}]_{\mathcal{F}} := \{n \in \mathbb{N} : r_n \leq s_n\} \in \mathcal{F} \quad (\text{B14})$$

The assumption of the Continuum Hypothesis (i.e. axiom A.12) implies that:

**Proposition B.4**

$({}^*\mathbb{R}_{\mathcal{F}_1}, +, \cdot, \leq)$  is isomorphic to  $({}^*\mathbb{R}_{\mathcal{F}_2}, +, \cdot, \leq) \quad \forall \mathcal{F}_1, \mathcal{F}_2 \in NPU(\mathbb{N})$

Proposition B.4 allows to give the following:

**Definition B.10**

*hyperreal number system of Nonstandard Analysis:*

$$({}^*\mathbb{R}, +, \cdot, \leq) := ({}^*\mathbb{R}_{\mathcal{F}}, +, \cdot, \leq) \quad \mathcal{F} \in NPU(\mathbb{N}) \quad (\text{B15})$$

Given  $x \in {}^*\mathbb{R}$ :

**Definition B.11**

$$|x| := \begin{cases} x, & \text{if } x \geq 0; \\ -x, & \text{if } x < 0. \end{cases} \quad (\text{B16})$$

**Definition B.12**

*x is infinitesimal:*

$$|x| < \epsilon \quad \forall \epsilon \in (0, +\infty) \quad (\text{B17})$$

**Definition B.13**

*x is limited:*

$$\exists r \in \mathbb{R} : |x| < r \quad (\text{B18})$$

**Definition B.14**

*x is unlimited:*

$$|x| > r \quad \forall r \in (0, +\infty) \quad (\text{B19})$$

Given  $x_1, x_2 \in {}^*\mathbb{R}$ :

**Definition B.15**

*$x_1$  is infinitely closed to  $x_2$ :*

$$x_1 \simeq x_2 := x_1 - x_2 \text{ is infinitesimal} \quad (\text{B20})$$

It can be easily verified that:

**Proposition B.5**

$\simeq$  is an equivalence relation over  ${}^*\mathbb{R}$ .

Given  $x \in {}^*\mathbb{R}$ :

**Definition B.16**

*halo of x:*

$$\text{hal}(x) := \{y \in {}^*\mathbb{R} : x \simeq y\} \quad (\text{B21})$$

Clearly the set of infinitesimal hyperreals is nothing but  $\text{hal}(0)$ .



**Proposition B.6**

1.

$$[\bar{r}]_{\mathcal{F}} \in \text{hal}(0) \quad \forall \bar{r} = \{r_n\}_{n \in \mathbb{N}} \in \mathbb{R}^{\mathbb{N}} : \lim_{n \rightarrow +\infty} r_n = 0 \quad (\text{B22})$$

2.

$$[\bar{r}]_{\mathcal{F}} \text{ is unlimited} \quad \forall \bar{r} = \{r_n\}_{n \in \mathbb{N}} \in \mathbb{R}^{\mathbb{N}} : \lim_{n \rightarrow +\infty} r_n = +\infty \quad (\text{B23})$$

**Remark B.1**

Nonstandard Analysis is often criticized for its claimed nonconstructive nature.

Alain Connes, for instance, claims in [49] that no element of  ${}^*\mathbb{R} - \mathbb{R}$  "can be exhibited".

So a short confutation of Connes' claim based on proposition B.6 could simply be the exhibition of the infinitesimal hyperreal  $[\{\frac{1}{n}\}_{n \in \mathbb{N}}]_{\mathcal{F}}$  or of the unlimited hyperreal  $[\{n\}_{n \in \mathbb{N}}]_{\mathcal{F}}$ .

A more detailed analysis of Connes' statement requires, anyway, a precise definition of the locution "exhibiting a mathematical object" and consequentially naturally leads us to the issue of furnishing a precise definition of the term "constructive".

As we have implicitly done in the remark A.2 and in the remark A.3 we will say that the proposition stating the existence of a mathematical object  $x$  is constructive whether it contains also the explicit definition of an algorithm, i.e. (assuming Church Thesis) a partial recursive function [55], [56], computing  $x$ .

With this regard the definition B.10 of the hyperreals as  $\sim_{\mathcal{F}}$ -equivalence classes of real sequences is as much constructive as the definition A.32 of the reals belonging to the interval  $[0, 1]$  as  $\sim_{\mathbb{R}}$ -equivalence classes of binary sequences.

The fact that the former is based on the nonconstructive Axiom of Choice is balanced by the fact the latter is (implicitly) based on the nonconstructive Axiom of Foundation.

Given  $x \in {}^*\mathbb{R}$  limited:

**Proposition B.7**

$$\exists ! st(x) \in \mathbb{R} : st(x) \simeq x \quad (\text{B24})$$

**Proposition B.8**

Proposition B.7 is equivalent to the Dedekind completeness of  $\mathbb{R}$

$st(x)$  is called the standard part of the limited hyperreal  $x$ .

Given a set  $A \in \mathcal{P}(\mathbb{R})$ :

**Definition B.17**

*enlargement of  $A$ :*

$${}^*A := \{[\bar{r}]_{\mathcal{F}} \in {}^*\mathbb{R} : \{n \in \mathbb{N} : r_n \in A\} \in \mathcal{F}\} \quad (\text{B25})$$

**Example B.1**

${}^*\mathbb{N}$  is usually called the set of all hypernatural numbers,  ${}^*\mathbb{Z}$  is usually called the set of all hyperinteger numbers and  ${}^*\mathbb{Q}$  is usually called the set of all hyperrational numbers.

Given a set  $S$ :

**Definition B.18**

*$S$  is hyperfinite:*

$$\exists n \in {}^*\mathbb{N} : S = \{k \in {}^*\mathbb{N} : k \leq n\} \quad (\text{B26})$$

**Remark B.2**

Let us remark that clearly, according to definition A.47, an hyperfinite set is infinite.

Given  $a, b \in \mathbb{R} : a < b$  and an hypernatural  $n \in {}^*\mathbb{N}$ :

### Definition B.19

*n*-sliced interval between  $a$  and  $b$ :

$$[a, b]_n := \{a + k \cdot \frac{b-a}{n} \mid k \in \{j \in \mathbb{N} : j \leq n\}\} \quad (\text{B27})$$

### Definition B.20

$[a, b]_n$  is an hyperfinite interval:

$$n \in {}^*\mathbb{N} - \mathbb{N} \quad (\text{B28})$$

### Remark B.3

Let us remark that clearly, according to definition A.47, an hyperfinite interval is infinite.

In order to introduce some more advanced technique of Nonstandard Analysis it is useful to introduce some new set-theoretic notion.

Given a set  $S$  and  $n \in \mathbb{N}$ :

### Definition B.21

$n^{th}$  cumulative power set of  $S$ :

$$\mathbb{U}_0(S) := S \quad (\text{B29})$$

$$\mathbb{U}_n(S) := \mathbb{U}_{n-1}(S) \cup \mathcal{P}(\mathbb{U}_{n-1}(S)) \quad (\text{B30})$$

The cumulative power sets of finite sets may be computed through the following Mathematica [57] code:

```
$RecursionLimit=Infinity;

<<DiscreteMath`Combinatorica`

powerset[x_] := LexicographicSubsets[x]

cumulativepowerset[x_, n_] :=
  If[n == 0, x, Union[cumulativepowerset[x, n - 1], powerset[cumulativepowerset[x, n - 1]]]]
```

### Example B.2

Let us compute the first cumulative power sets of the empty set. We obtain that:

[illegible]

and so on.

**Definition B.22**

*superstructure over  $S$ :*

$$\mathbb{U}(S) := \cup_{n \in \mathbb{N}} \mathbb{U}_n(S) \quad (\text{B31})$$

Substantially every mathematical object needed to study  $S$  is an element of  $\mathbb{U}(S)$ .

For instance:

**Proposition B.9**

1. the set of all topologies on  $S$  is an element of  $\mathbb{U}_3(S)$
2. the set of all measures on  $S$  is an element of  $\mathbb{U}_5(S)$
3. the set of all metrics on  $S$  is an element of  $\mathbb{U}_6(S)$

Given  $x \in \mathbb{U}(S)$ :

**Definition B.23**

*rank of  $x$ :*

$$\text{rank}(x) := \min\{n \in \mathbb{N} : x \in \mathbb{U}_n(S)\} \quad (\text{B32})$$

The basic mathematical object of Nonstandard Analysis is then a suitably defined extension map  $\star : \mathbb{U}(\mathbb{R}) \mapsto \mathbb{U}(\star\mathbb{R})$ .

Given  $x \in \mathbb{U}(\star\mathbb{R})$ :

**Definition B.24**

*$x$  is internal:*

$$x \in \star\mathbb{U}(\mathbb{R}) \quad (\text{B33})$$

**Definition B.25**

*$x$  is external:*

$$x \notin \star\mathbb{U}(\mathbb{R}) \quad (\text{B34})$$

The corner-stone of Nonstandard Analysis is the following:

**Proposition B.10**

*Transfer Principle:*

$$(\phi \text{ holds in } \mathbb{U}(\mathbb{R})) \Leftrightarrow \star\phi \text{ holds in } \star\mathbb{U}(\mathbb{R}) \quad \forall \phi \text{ bounded-quantifier statement}$$

A consequence of Proposition B.10 is the following:

**Proposition B.11**

*Conservative Property of Nonstandard Analysis:*

Every theorem about  $\mathbb{U}(\mathbb{R})$  that can be proved resorting to Nonstandard Analysis (i.e. by using elements of  $\mathbb{U}(\star\mathbb{R})$ ) can be also proved without resorting to Nonstandard Analysis.

Proposition B.11 could lead to think that Nonstandard Analysis is useless.

It has to be stressed, with this regard, that though every theorem about  $\mathbb{U}(\mathbb{R})$  that can be proved resorting to Nonstandard Analysis can also be proved without resorting to it, the complexity of a proof resorting to Nonstandard Analysis may be lower than the complexity of a proof non resorting to it.

Let us now introduce some notion of Nonstandard Topology.

Given  $M \in \mathbb{U}(\mathbb{R})$  let us recall first of all that:

**Definition B.26**

*topology over M:*

$$\mathcal{T} \subseteq \mathcal{P}(M) :$$

•

$$\emptyset, S \in \mathcal{T} \tag{B35}$$

•

$$O_1, O_2 \in \mathcal{T} \Rightarrow O_1 \cap O_2 \in \mathcal{T} \tag{B36}$$

•

$$O_i \in \mathcal{T} \forall i \in I \Rightarrow \cup_{i \in I} O_i \in \mathcal{T} \tag{B37}$$

We will denote the set of all the topologies over M by TOP(M).

By proposition B.9 it follows that:

**Proposition B.12**

$$TOP(M) \in \mathbb{U}_{rank(M)+3}(\mathbb{R}) \tag{B38}$$

Given  $\mathcal{T} \in TOP(M)$  and  $q_1, q_2 \in M$ :

**Definition B.27**

$$q_1 \curlyvee q_2 := O_1 \cap O_2 \neq \emptyset \quad \forall O_1, O_2 \in \mathcal{T} : q_1 \in O_1 \wedge q_2 \in O_2 \tag{B39}$$

**Definition B.28**

$\mathcal{T}$  is Hausdorff:

$$\neg(q_1 \curlyvee q_2) \quad \forall q_1, q_2 \in M \tag{B40}$$

Given  $q_1 \in M$ :

**Definition B.29**

*halo of  $q_1$ :*

$$hal(q_1) := \cap_{q_1 \in O \in \mathcal{T}} {}^*O \tag{B41}$$

Then:

**Proposition B.13**

$$\mathcal{T} \text{ is Hausdorff} \Leftrightarrow hal(q_1) \cap hal(q_2) = \emptyset \quad \forall q_1, q_2 \in M : q_1 \neq q_2, \forall \mathcal{T} \in TOP(M) \tag{B42}$$

Given  $q_2 \in {}^*M$ :

**Definition B.30**

$q_2$  is infinitely closed to  $q_1$ :

$$q_2 \simeq q_1 := q_2 \in hal(q_1) \tag{B43}$$

**Definition B.31**

*nearstandard points of  $M$ :*

$$ns(M) := \{q_1 \in {}^*M : (\exists q_2 \in M : q_1 \simeq q_2)\} \quad (\text{B44})$$

Then:

**Proposition B.14**

HP:

$$\mathcal{T} \in TOP(M) \quad (\text{B45})$$

TH:

$$\mathcal{T} \text{ is Hausdorff} \Rightarrow \forall q \in ns(M) \exists ! st(q) \in M : st(q) \simeq q \quad (\text{B46})$$

**Remark B.4**

Let us consider the particular case in which  $M := \mathbb{R}$  while  $\mathcal{T}_{natural}$  is the natural topology over  $\mathbb{R}$ , i.e. the topology induced by the metric  $d_{euclidean}$  induced by the euclidean riemannian metric  $\delta$  over  $\mathbb{R}$ .

Then:

$$ns(\mathbb{R}) = \{x \in {}^*\mathbb{R} : x \text{ is limited}\} \quad (\text{B47})$$

Hence, in this case, proposition B.14 reduces to proposition B.7 that, by proposition B.8, we know to be equivalent to the Dedekind completeness of  $\mathbb{R}$ .

Consequentially one obtains a different topological viewpoint on the Dedekind completeness of  $\mathbb{R}$  that deeply links it to the Hausdorffness of  $\mathcal{T}_{natural}$ .

## APPENDIX C: NONSTANDARD ANALYSIS GOING OUTSIDE ZFC: INTERNAL SET THEORY AND ITS BUGS

Edward Nelson [67] has introduced an alternative formulation of a part of Nonstandard Analysis in which the same formal system ZFC axiomatizing Set Theory is extended to a new formal system, called Internal Set Theory (shortened as IST), obtained from ZFC by:

1. adding a new undefined unary predicate "standard"
2. defining an *internal formula* of Internal Set Theory as a formula of ZFC not containing the new predicate "standard"
3. defining an *external formula* of Internal Set Theory as a not internal formula
4. adding to the axioms of ZFC three new axioms (the Axiom of Idealization, the Axiom of Standardization and the Axiom of Transfer)

### Definition C.1

*Internal Set Theory (IST):*

the formal system obtained augmenting ZFC with the following:

#### AXIOM C.1

*Axiom of Transfer:*

HP:

$A(x, t_1, \dots, t_k)$  internal formula with free variables  $x, t_1, \dots, t_k$  and no other free variables

TH:

$$\forall^{st} t_1 \dots \forall^{st} t_k (\forall^{st} x A(x, t_1, \dots, t_k) \Rightarrow \forall x A(x, t_1, \dots, t_k)) \quad (C1)$$

#### AXIOM C.2

*Axiom of Idealization:*

HP:

$B(x, y)$  internal formula with free variables  $x, y$  and possibly other free variables

TH:

$$\forall^{st} fin z \exists x, y \in z : B(x, y) \Leftrightarrow \exists x : \forall^{st} y B(x, y) \quad (C2)$$

### AXIOM C.3

*Axiom of Standardization:*

HP:

$C(z)$  formula, internal or external, with free variable  $z$  and possibly other free variables

TH:

$$\forall^{st} x \exists^{st} y : \forall^{st} z (z \in y \Leftrightarrow z \in x \wedge C(z)) \quad (C3)$$

where we have adopted the following abbreviations:

$$\forall^{st} x := \forall x : x \text{ standard} \quad (C4)$$

$$\exists^{st} x := \exists x : x \text{ standard} \quad (C5)$$

$$\forall^{fin} x := \forall x : x \text{ finite} \quad (C6)$$

$$\exists^{fin} x := \exists x : x \text{ finite} \quad (C7)$$

#### Remark C.1

It is important to stress that the fact that the underlying formal system is different results in that an internal formula  $\phi_{IST}$  of IST has a meaning that is different from the meaning that the same formula  $\phi_{ZFC}$  has within ZFC.

This implies that the same formal definition of a mathematical object  $x$  results in different mathematical objects according to whether it is considered within ZFC or within IST;

let us denote by  $x_{ZFC}$  and by  $x_{IST}$  a mathematical object  $x$  considered within, respectively, the formal system ZFC and IST.

#### Remark C.2

Since IST is an extension of ZFC it follows that if  $\phi_{ZFC}$  is a theorem of ZFC then  $\phi_{IST}$  is an internal theorem of IST.

It is important, anyway, to stress that  $\phi_{IST} \neq \phi_{ZFC}$ .

For instance the fact that theorem A.3 holds within ZFC implies that a corresponding theorem holds within IST:

#### Theorem C.1

*Existence and unicity of IST's naturals*

There exists exactly one set  $\mathbb{N}_{IST}$  such that:

1.

$$\emptyset \in \mathbb{N}_{IST} \quad (C8)$$

2.

$$Suc(x) \in \mathbb{N}_{IST} \quad \forall x \in \mathbb{N}_{IST} \quad (C9)$$

where, as in the Axiom of Infinity (i.e axiom A.8),  $Suc(x) := x \cup \{x\}$ .

3. if  $K$  is any set that satisfies eq. C8 and eq. C9 then  $\mathbb{N}_{IST} \subset K$ .

PROOF:

By the Axiom of Infinity (i.e axiom A.8) there exists at least one set  $X$  satisfying eq. C8 and eq. C9.

Let:

$$\mathcal{F} := \{Y \in \mathcal{P}(X) : \emptyset \in Y \wedge (Suc(x) \in Y \forall x \in Y)\} \quad (C10)$$

$$\mathbb{N}_{IST} := \bigcap_{Y \in \mathcal{F}} Y \quad (C11)$$

It is easy to see that the intersection of any nonempty family of sets satisfying eq. C8 and eq. C9 still satisfies eq. C8 and eq. C9.

Let  $K$  be a set that satisfies eq. C8 and eq. C9; then:

$$X \cap K \in \mathcal{F} \quad (C12)$$

and:

$$\mathbb{N}_{IST} := \bigcap_{Y \in \mathcal{F}} Y \subset X \cap K \subset K \quad (C13)$$

■

It is important anyway to stress that theorem A.3 and theorem C.1 have different meaning:

the former gives an implicit definition of the set  $\mathbb{N}_{ZFC}$  while the latter gives an implicit definition of the set  $\mathbb{N}_{IST}$  where:

$$\mathbb{N}_{ZFC} \neq \mathbb{N}_{IST} \quad (C14)$$

Consequently the chain of mathematical definitions given in the section A gives rise to sets  $\mathbb{Z}_{IST}$ ,  $\mathbb{Q}_{IST}$ ,  $\mathbb{R}_{IST}$  such that:

$$\mathbb{Z}_{ZFC} \neq \mathbb{Z}_{IST} \quad (C15)$$

$$\mathbb{Q}_{ZFC} \neq \mathbb{Q}_{IST} \quad (C16)$$

$$\mathbb{R}_{ZFC} \neq \mathbb{R}_{IST} \quad (C17)$$

Actually it may be proved that:

**Theorem C.2**

1.

$$\mathbb{N}_{IST} = (*\mathbb{N})_{ZFC} \quad (C18)$$

2.

$$\mathbb{Z}_{IST} = (*\mathbb{Z})_{ZFC} \quad (C19)$$

3.

$$\mathbb{Q}_{IST} = (*\mathbb{Q})_{ZFC} \quad (C20)$$

4.

$$\mathbb{R}_{IST} = (*\mathbb{R})_{ZFC} \quad (C21)$$

The great conceptual bug of Internal Set Theory consists in that:



**Theorem C.3**

$\mathbb{N}_{ZFC}$  cannot be defined within IST

PROOF:

Let us assume ad absurdum that  $(\mathbb{N})_{ZFC}$  can be defined within IST.

Then it satisfies the conditions eq. C8 and eq. C9 and hence, by theorem C.1, one has that:

$$(\mathbb{N})_{IST} \subset (\mathbb{N})_{ZFC} \tag{C22}$$

that is in contradiction with theorem C.2 ■

**Remark C.3**

It is important to remark that within IST given a set  $S$  and a unary predicate  $p$  one can use the Comprehension Axiom (i.e. axiom A.4) to define the set  $S_p := \{x \in S : p(x)\}$  if and only if  $p$  is internal.

Since the predicate:

$$p_{standard}(x) := x \text{ is standard} \tag{C23}$$

is not internal it follows that it cannot be used to define the set  $S_{p_{standard}}$ .

This applies in particular for  $S \in \{\mathbb{N}, \mathbb{Z}, \mathbb{Q}, \mathbb{R}\}$ .

**Remark C.4**

Theorem C.3 implies that every branch of Mathematics, such as Arithmetics or Classical Recursion Theory [55], [56], that cannot be formulated without introducing the set  $(\mathbb{N})_{ZFC}$  cannot be formulated within Internal Set Theory (see the third chapter "Theories of internal sets" of [68] and in particular the section 3.6c "Three "myths" of IST").

## APPENDIX D: LOEB PROBABILITY SPACES

In a remarkable paper [69] Peter Loeb introduced a very rich class of standard measure spaces constructed using Nonstandard Analysis (in its orthodox formulation given within ZFC+CH presented in section B). In the later decade Loeb Measures has demonstrated to be a very powerful tool in the framework of Classical Probability Theory [65], [40].

Let  $\Omega \in {}^*\mathbb{U}(\mathbb{R})$  be an internal set,  $\mathcal{A} \subseteq \mathcal{P}(\Omega)$  be an algebra, and  $\mu : \mathcal{A} \rightarrow {}^*[0, \infty)$  be an internal finitely additive measure on  $\mathcal{A}$  normalized to one, i.e:

$$\mu(A_1 \cup A_2) = \mu(A_1) + \mu(A_2) \quad \forall A_1, A_2 \in \mathcal{A} : A_1 \cap A_2 = \emptyset \quad (\text{D1})$$

$$\mu(\Omega) = 1 \quad (\text{D2})$$

Given  $B \in \mathcal{P}(\Omega)$  (not necessarily internal):

### Definition D.1

$B$  is a Loeb  $\mu$ -null set:

$$\forall \epsilon \in (0, +\infty), \exists A \in \mathcal{A} : B \subseteq A \wedge \mu(A) < \epsilon \quad (\text{D3})$$

### Definition D.2

Loeb  $\sigma$ -algebra w.r.t.  $\mathcal{A}$  and  $\mu$ :

$$L(\mathcal{A}, \mu) := \{B \in \mathcal{P}(\Omega) : (\exists A \in \mathcal{A} : A \triangle B \text{ is a Loeb } \mu\text{-null set})\} \quad (\text{D4})$$

where:

$$A \triangle B := (A - B) \cup (B - A) \quad (\text{D5})$$

is the symmetric difference of  $A$  and  $B$ .

### Definition D.3

Loeb probability space w.r.t.  $(\Omega, \mathcal{A}, \mu)$ :

the classical probability space  $(\Omega, L(\mathcal{A}, \mu), \mu_L)$ , where  $\mu_L : L(\mathcal{A}) \mapsto [0, 1]$ , said a Loeb probability measure, is defined as:

$$\mu_L(A) := st(\mu(A)) \quad A \in L(\mathcal{A}) \quad (\text{D6})$$

### Remark D.1

Let us remark that a Loeb probability space is a classical probability space in the sense of the Kolmogorov axiomatization [70], [71].

A particularly important example of a Loeb probability measure is the Loeb counting measure we are going to introduce.

Given  $n \in {}^*\mathbb{N} - \mathbb{N}$  let us consider the hyperfinite set:

$$\Omega := \{j \in {}^*\mathbb{N} : 1 \leq j \leq n\} \quad (\text{D7})$$

and the map  $\nu : {}^*\mathcal{P}(\Omega) \mapsto {}^*[0, 1]$ :

$$\nu(A) := \frac{{}^*|A|}{n} \quad (\text{D8})$$

where  ${}^*|\cdot|$  is the extension to  ${}^*\mathbb{U}(\mathbb{R})$  of the function  $|\cdot|$  that gives the cardinality of finite sets.

### Definition D.4

$n^{\text{th}}$  counting Loeb probability space

the Loeb probability space  $(\Omega, L[{}^*\mathcal{P}(\Omega), \nu], \nu_L)$ .

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